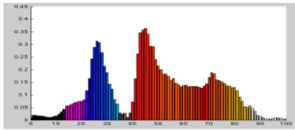


Signals and Systems
Lecture 15-Fourier Series Representation of Periodic Signals (Part 3)

Outline
- Examples.
- Convergence properties of Fourier series.
- Dirichlet Conditions (convergence of FS).

Examples

Example 4:

Find the Fourier series coefficients (exponential form) for the Signal:

$$x(t) = 3 + 8\cos^2(5\pi t) + 6\sin(15\pi t)$$

Remark

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

Solution:

$$x(t) = 3 + 8\cos^2(5\pi t) + 6\sin(15\pi t)$$

$$= 3 + 8\left[\frac{1}{2} + \frac{1}{2}\cos(10\pi t)\right] + 6\sin(15\pi t)$$

$$= 7 + 4\cos(10\pi t) + 6\sin(15\pi t)$$

$$= 7 + 2e^{j10\pi t} + 2e^{-j10\pi t} + \frac{3}{j}e^{j15\pi t} - \frac{3}{j}e^{-j15\pi t}$$

The fundamental frequency is

$$\omega_0 = 5\pi \text{ and}$$

$$d_0 = 7, \quad d_2 = d_{-2} = 2, \quad d_3 = d^*_{-3} = \frac{3}{j}, \quad d_{-3} = -\frac{3}{j}$$

$$d_k = 0 \quad for \quad k \neq 0, \pm 2, \pm 3$$

Example 5:

Find the exponential Fourier series for the signal in Example 3 (previous lecture).

Solution:

In this case

$$To = \pi, \quad \omega o = \frac{2\pi}{T_0} = 2$$

$$x(t) = \sum_{k=-\infty}^{\infty} D_k \,.\, e^{j2kt}$$

$$D_k = \frac{1}{T_0} \,.\, \int_{<T_0>} x(t).e^{-j2kt}dt = \frac{1}{\pi} \,.\, \int_0^{\pi} e^{-\frac{t}{2}}.e^{-j2kt}dt$$

$$= \frac{1}{\pi} \cdot \int_{0}^{\pi} e^{-\left(\frac{1}{2}+j2k\right)t} dt$$

$$= \frac{-1}{\pi\left(\frac{1}{2}+j2k\right)} \cdot \left[e^{-\left(\frac{1}{2}+j2k\right)t}\right]_0^{\pi} = \frac{0.504}{1+j4k}$$

$$x(t) = 0.504 \sum_{k=-\infty}^{\infty} \frac{1}{1+j4k} \cdot e^{j2kt}$$

$$= 0.504\left[1 + \frac{1}{1+j4} e^{j2t} + \frac{1}{1+j8} e^{j4t} + \frac{1}{1+j12} e^{j6t} + \cdots + \frac{1}{1-j4} e^{-j2t} + \frac{1}{1-j8} e^{-j4t} + \frac{1}{1-j12} e^{-j6t} + \cdots\right]$$

Observe that the coefficients D_k are complex, Moreover D_k and D_{-k} are conjugate as expected.

Exponential Fourier spectra:

$$D_0 = 0.504$$

$$D_1 = \frac{0.504}{1+j4} = 0.122.e^{-j75.96^\circ} \Rightarrow |D_1| = 0.122, < D_1 = -75.96^\circ$$

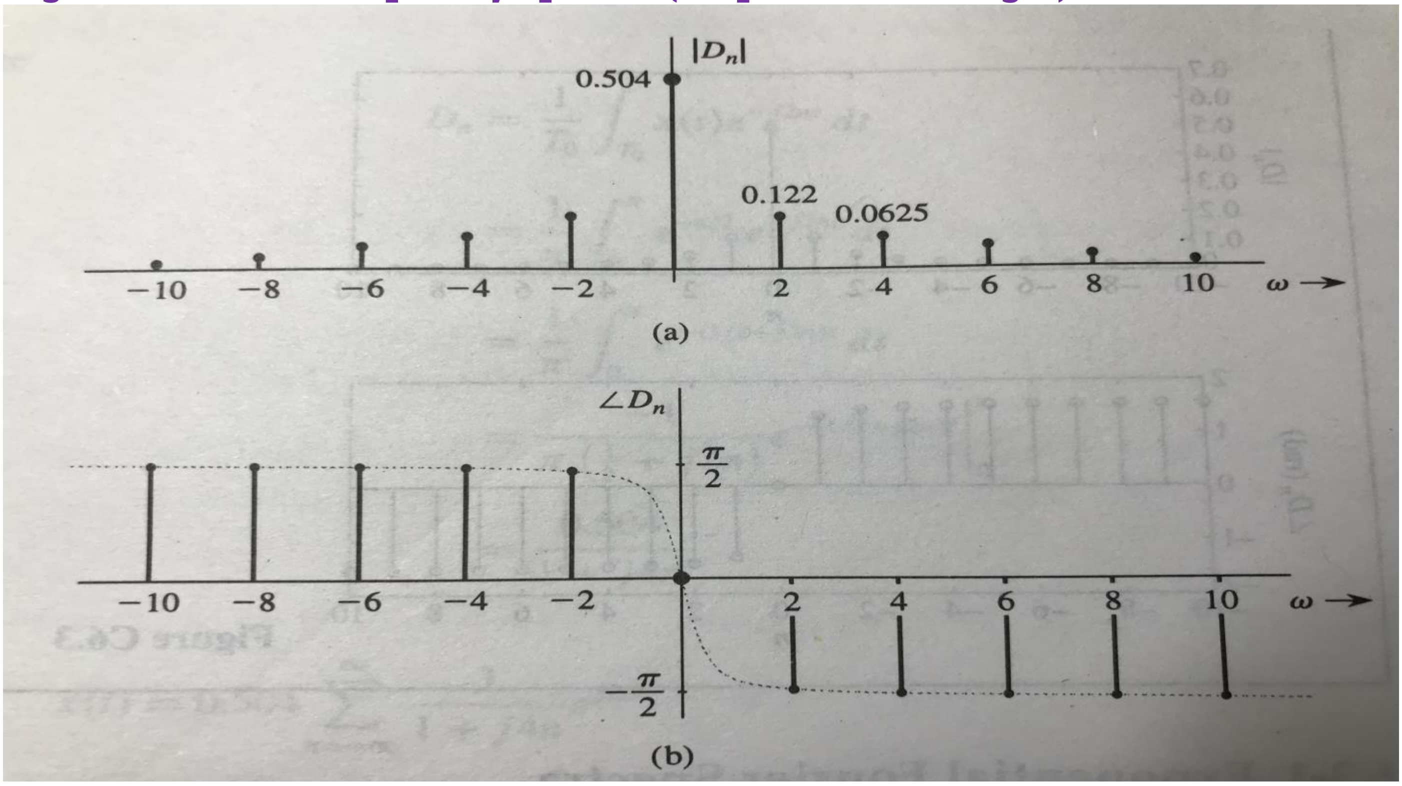
$$D_{-1} = \frac{0.504}{1-j4} = 0.122.e^{j75.96^\circ} \Rightarrow |D_{-1}| = 0.122, < D_{-1} = -75.96^\circ$$

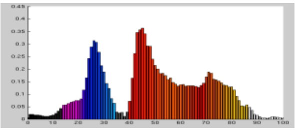
$$D_2 = \frac{0.504}{1+8J} = 0.0625.e^{-j82.87^\circ} \Rightarrow |D_2| = 0.0625, < D_2 = -82.87^\circ$$

$$D_{-2} = \frac{0.504}{1-8j} = 0.0625.e^{j82.87^\circ} \Rightarrow |D_{-2}| = 0.0625, < D_{-2} = 82.87^\circ$$

And so on ....

Figure shows the frequency spectra (amplitude and angle)

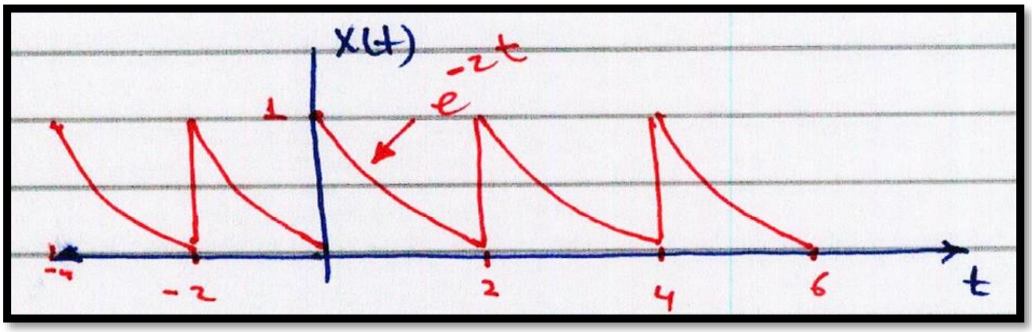




Example 6:

Direct Calculation of FS coefficients:

Determine the FS coefficients for the signal $x(t)$ depicted in the following figure.



Solution:

The period of $x(t)$ is $T = 2$, so $\omega_0 = \frac{2\pi}{2} = \pi$

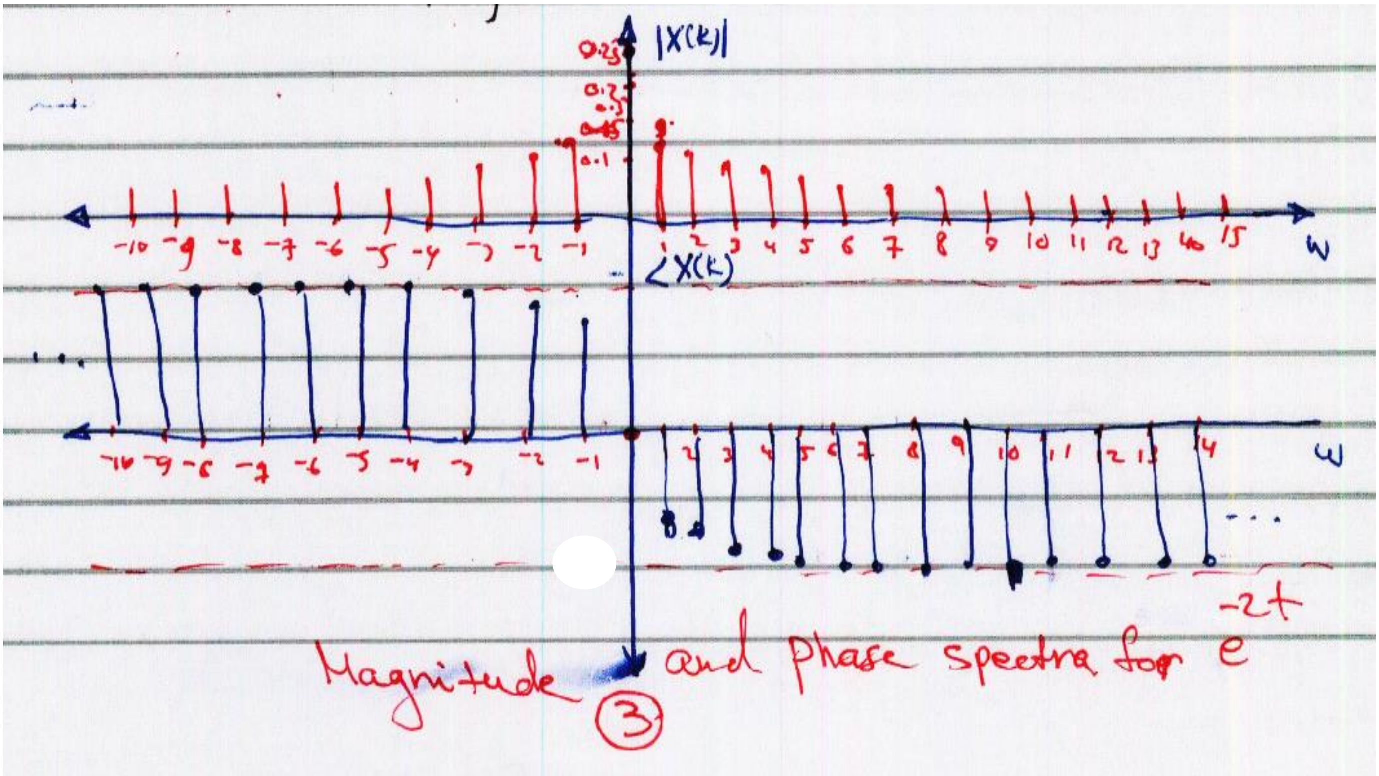
On the interval $0 \le t \le 2$ one period of $x(t)$ is expressed as $x(t) = e^{-2t}$,so

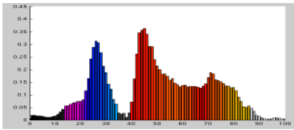
$$X(k) = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}dt \Rightarrow$$

$$X(k) = \frac{1}{2}\int_0^2 e^{-2t}.e^{-jk\pi t}dt = \frac{1}{2}\int_0^2 e^{-(2+jk\pi)t}dt$$

$$X(k) = -\frac{1}{2(2+jk\pi)}.\left[e^{-(2+jk\pi)t}\right]_0^2 = \frac{1}{(4+jk2\pi)}(1 - e^{-4}.e^{-jk2\pi})$$

$$= \frac{1-e^{-4}}{(4+jk2\pi)}, \qquad \textit{since } e^{-jk2\pi} = 1$$





Convergence properties of Fourier series

- **Since a Fourier series can have an infinite number of terms, and an infinite sum may or may not converge, we consider the issue of convergence.**
- **Let $x_N(t)$ denote the Fourier series truncated after N_{th} harmonics:**

$$x_N(t) = \sum_{k=-N}^{N} D_k \, . \, e^{jk\omega_0 t} \Rightarrow (partial\ sum)$$

- **The error in approximating $x(t)$ by $x_N(t)$is given by**

$$e_N(t) = x(t) - x_N(t)$$

- **and the corresponding mean-squared error (MSE) (i.e. energy of the error is given by**

$$E_N = \frac{1}{T} \int_{\langle T \rangle} |e_N(t)|^2 \, dt$$

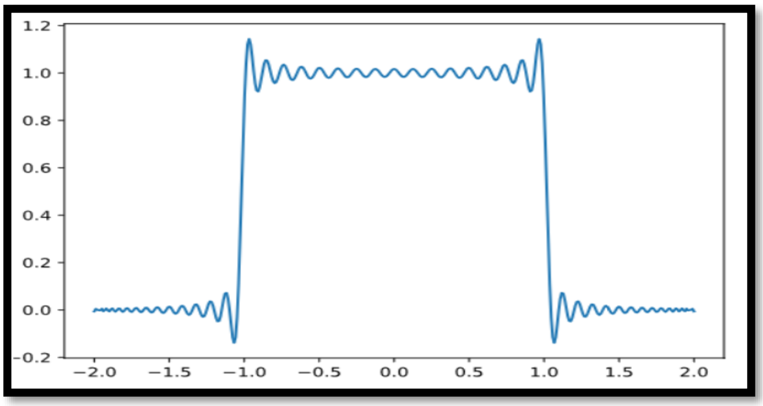
> $e_N(t)$- **The difference between the original signal and the partial sum.**

- **If a periodic signal $x(t)$ has finite energy in a single period**

$$\int_{\langle T \rangle} |x(t)|^2 \, dt < \infty$$

Then, the Fourier series converges in the MSE Sense.

- **In square wave signal (video lecture 7, Minute: 36): Increasing the number of terms up to 101, we notice that the signal is approximated and sharp edges is not good "Gibbs phenomenon".**



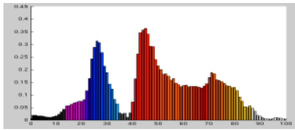
- **The issue of convergence of Fourier series: Does the error $e_N(t)$ decrease as N increases?**
  - **If**

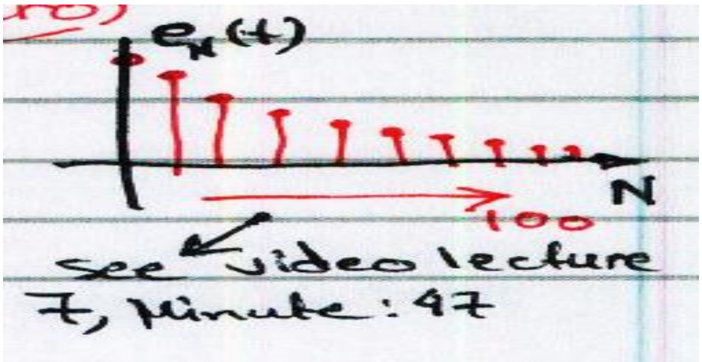
$$\int_{<T>} |x(t)|^2 \, dt < \infty$$

Then

$$\int_{<T>} |e_N(t)|^2 \, dt \to 0 \quad as \quad N \to \infty$$

Except at discontinuities





Dirichlet Conditions (convergence of FS)

- **The Fourier Series is convergent if the signal $x(t)$ Satisfies some conditions (Dirichlet conditions):**

1) ***Single value property*: $x(t)$ must have only one value at any time instant within the T_0.**

2) ***Finite discontinuities*: $x(t)$ should have at most finite number of discontinuities in the interval T_0.**

3) ***Finite peaks*: The signal $x(t)$ should have finite number of maxima and minima in the interval T_0.**

4) ***Absolute integrability* : the signal should be absolutely integrable:**

$$\int_{<T>} x(t)dt < \infty$$

- **Above conditions are sufficient but not necessary conditions for Fourier series representation.**
- **Most of physical signals satisfy above conditions.**
- **The points of discontinuities, $x(t)$ converges to the value "Midway" between the two values of $x(t)$ on either sides of the discontinuity.**